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RICCATI REPRESENTATION FOR ELEMENTS IN $H^{-1}(\mathbb{T})$ AND ITS APPLICATIONS

Thomas Kappeler* and Peter Topalov

ABSTRACT. This paper is concerned with the spectral properties of the Schrödinger operator $L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$ with periodic potential q from the Sobolev space $H^{-1}(\mathbb{T})$. We obtain asymptotic formulas and a priori estimates for the periodic and Dirichlet eigenvalues which generalize known results for the case of potentials $q \in L_0^2(\mathbb{T})$. The key idea is to reduce the problem to a known one – the spectrum of the impedance operator – via a nonlinear analytic isomorphism of the Sobolev spaces $H_0^{-1}(\mathbb{T})$ and $L_0^2(\mathbb{T})$.

1. Introduction

The present paper is devoted to the spectral properties of the Schrödinger operator $L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$ with “singular” potential q from the Sobolev space $H^{-1}(\mathbb{T})$ viewed as an unbounded operator on $H^{-1}(\mathbb{T})$. Our approach is based on a nonlinear representation - clearly of independent interest - of the elements of the Sobolev space $H^{-1}(\mathbb{T})$ in terms of a unique function from $L_0^2(\mathbb{T})$, referred to as Riccati representation.

Let $\rho \in H^1(\mathbb{T})$ with $\min_{x \in \mathbb{T}} \rho(x) > 0$ and define $r \stackrel{\text{def}}{=} \rho'/\rho$. By the transformation $y \stackrel{\text{def}}{=} \rho \tilde{y}$ the equation $-\tilde{y}'' - 2r\tilde{y}' = \lambda \tilde{y}$ becomes the Schrödinger equation $-y'' + qy = \lambda y$ with $q = r' + r^2$. It was conjectured in [3] that the spectral properties of the Schrödinger operator L_q could be deduced from the spectral

2002 *Mathematics Subject Classification*: 35L05, 34L15, 35D05, 35Q53

Key words: Hill operator, Riccati transform, Birkhoff coordinates, KdV

* Supported in part by European Research Training Network HPRN-CT-1999-00118

properties of the impedance operator $T_r(u) \stackrel{\text{def}}{=} -(\rho^2 u')'/\rho^2$, whose spectral theory is well-developed ([13, 1, 2, 9]). In the present paper we prove that for any $\alpha \geq 0$ the Riccati map $r \mapsto r' + r^2 - \|r\|^2$ maps $H_0^\alpha(\mathbb{T})$ onto $H_0^{\alpha-1}(\mathbb{T})$ where $\|r\|^2 = \int_0^1 r^2(x) dx$ and $H_0^\alpha(\mathbb{T}) \stackrel{\text{def}}{=} \{f \in H^\alpha(\mathbb{T}) \mid \int_0^1 f dx = 0\}$. In fact it is a real-analytic isomorphism between $H_0^\alpha(\mathbb{T})$ and $H_0^{\alpha-1}(\mathbb{T})$ (see Section 3.). Our proof is elementary and based on the spectral properties of the Schrödinger operator. In particular, the properties of the first eigenvalue $\lambda_0(q)$ of the Schrödinger operator and the corresponding normalized eigenfunction $f_0(q)$ are essential for our approach.

The main result of this paper is summarized in the following theorem.

Theorem 1. *The Riccati map $R : L_0^2(\mathbb{T}) \rightarrow H_0^{-1}(\mathbb{T})$ is a real analytic isomorphism. For any $r \in L_0^2(\mathbb{T})$, the impedance operator T_r and the associated Schrödinger operator L_q with $q = R(r)$ have, up to a translation by $\|r\|^2$, the same periodic and Dirichlet spectrum. Moreover, $R^{-1}(q)$ is given by*

$$R^{-1}(q) = f'_0(\cdot, q)/f_0(\cdot, q)$$

where $f_0(\cdot, q)$ is an eigenfunction (in $H^1(\mathbb{T})$) corresponding to the first (lowest) periodic eigenvalue $\lambda_0(q)$ of L_q which can be proved to vanish nowhere.

The paper is organized as follows. In Section 2. we prove auxiliary results needed for the proof of Theorem 1 in Section 3.. Section 4. is devoted to applications of the Riccati representation needed in an essential way in subsequent work [4, 7]. In Section 4.1. we prove spectral results for the Hill operator L_q . The Dirichlet spectrum of L_q is considered in Section 4.2.. The results there generalize results of [17] to the case of singular potentials $q \in H^{-1}(\mathbb{T})$. Finally, we extend the notion of discriminant to potentials in $H^{-1}(\mathbb{T})$ and prove the isospectral invariance of the Riccati map (Section 4.4.). To make the paper self-contained we include Appendices A and B where we present results on the spectrum of T_r and L_q used in this paper.

The spectral theory of operators L_q with singular potentials $q \in H^\alpha(\mathbb{T})$, $\alpha > -1$, was developed in [3] and [5] and was partly motivated by the construction of action-angle coordinates on the phase space $H^\alpha(\mathbb{T})$ for the Korteweg-de Vries equation (KdV) – see the introduction in [3]. In [4] we extend the construction of action-angle coordinates for potentials in $H^{-1}(\mathbb{T})$ using the applications of Theorem 1 stated in Section 4.. In the subsequent paper [7] the Birkhoff coordinates are used to prove that KdV is well posed in $H^{-1}(\mathbb{T})$.

In the stage of finishing this paper we were informed by E. Korotyaev that at the same time but by different proofs he showed that the Riccati map is a

real analytic isomorphism. Being interested rather in spectral problems than in applications to KdV, he used this result to solve an inverse problem ([11]). Note that there are no results in [11] concerning the Dirichlet spectrum of L_q .

This paper is an abridged version of [6].

The following notations are used throughout the paper. Denote by $h^{\beta,n}$ ($\beta \in \mathbb{R}, n \in \mathbb{Z}$) the Hilbert space of sequences $\{x_k\}_{k \in \mathbb{Z}} \subset \mathbb{C}$ with finite norm $\|x\|_{\beta,n} \stackrel{\text{def}}{=} (\sum_{k \in \mathbb{Z}} \langle k+n \rangle^{2\beta} |x_k|^2)^{1/2}$, where $\langle s \rangle \stackrel{\text{def}}{=} |s|+1$. The scalar product in $h^{\beta,n}$ is given by $(x, y)_{\beta,n} \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} \langle k+n \rangle^{2\beta} x_k \bar{y}_k$. By definition, $h^\beta \stackrel{\text{def}}{=} h^{\beta,0}$.

By \mathbb{T}_l , $l > 0$, we denote the one-dimensional torus $\mathbb{T}_l \stackrel{\text{def}}{=} \mathbb{R}/l\mathbb{Z}$. The Sobolev spaces $H^m(\mathbb{T}_l)$ and $H^m[0,1]$, $m \in \mathbb{N}$, are defined by $H^m(\Omega) \stackrel{\text{def}}{=} \{f : \Omega \rightarrow \mathbb{R} \mid f^{(k)} \in L^2(\Omega), k = 0, 1, \dots, m\}$ where $f^{(k)}$ is the k 'th distributional derivative of the function f and Ω denotes the torus \mathbb{T}_l or the interval $[0,1]$ respectively. The scalar product in $H^m(\Omega)$ is defined by $(f, g)_m \stackrel{\text{def}}{=} \sum_{k=0}^m (f^{(k)}, g^{(k)})$ where (\cdot, \cdot) denotes the L^2 -scalar product. For real $\alpha \geq 0$ the Sobolev spaces $H^\alpha(\mathbb{T}_l)$ and $H^\alpha[0,1]$ can be defined in a standard manner, for example, by interpolation (see [14], Chapter 1). By definition, $H^{-\alpha}(\Omega)$ is the dual space of $H_c^\alpha(\Omega)$, i.e. $H^{-\alpha}(\Omega) \stackrel{\text{def}}{=} (H_c^\alpha(\Omega))'$ where $H_c^\alpha(\Omega)$ is the closure in $H^\alpha(\Omega)$ of the space $C_c^\infty(\Omega)$ of smooth functions with compact support in the interior of Ω . The norm of f in $H^\alpha(\Omega)$ is denoted by $\|f\|_\alpha$ and for $\alpha = 0$ we write $\|f\| = \|f\|_0$. The distributional derivative $\frac{d}{dx} : H^m(\Omega) \rightarrow H^{m-1}(\Omega)$, $m \in \mathbb{N}$, can be extended in a natural way to a bounded operator $\frac{d}{dx} : H^\alpha(\Omega) \rightarrow H^{\alpha-1}(\Omega)$ for arbitrary $\alpha \in \mathbb{R}$. By $H_0^\alpha(\mathbb{T}_l)$ we denote the linear subspace of elements $f \in H^\alpha(\mathbb{T}_l)$ with mean value zero, $[f] \stackrel{\text{def}}{=} \int_{\mathbb{T}_l} f dx = 0$. Note that the Sobolev spaces $H^\alpha(\mathbb{T}_l)$, $\alpha \in \mathbb{R}$, can be identified (up to an equivalence of the norms) with the space of Fourier series $\sum_{k \in \mathbb{Z}} \hat{f}_k e^{2i\pi kx/l}$ whose Fourier coefficients $\{\hat{f}_k\}_{k \in \mathbb{Z}}$ have finite h^α -norm.

2. L_q associated to T_r

For any given $r \in L_0^2(\mathbb{T})$ denote by T_r the impedance operator

$$(1) \quad T_r(u) \stackrel{\text{def}}{=} -\frac{1}{\rho^2}(\rho^2 u')' = -u'' - 2ru'$$

on $L^2(\mathbb{T}_2)$ with domain $\text{Dom}(T_r) = H^2(\mathbb{T}_2)$. Here ρ is the absolutely continuous, 1-periodic, positive function given by $\rho(x) \stackrel{\text{def}}{=} \exp\left(\int_0^x r(v) dv\right)$. In particular, $\rho \in H^1(\mathbb{T})$ and $\rho' = r\rho$. Note that T_r is an operator with compact resolvent, non-negative, and symmetric with respect to the inner product $(f, g)_\rho \stackrel{\text{def}}{=} \int_0^2 f g \rho^2 dx$

on $L^2(\mathbb{T}_2)$. Hence the spectrum $\mathbf{spec}(T_r)$ is discrete, real and non-negative. It turns out to be of the form $\mathbf{spec}(T_r) = \{0 = \tilde{\lambda}_0(r) < \tilde{\lambda}_1(r) \leq \tilde{\lambda}_2(r) \leq \dots\}$. The corresponding eigenspaces are of finite dimension, and $\tilde{\lambda}_k(r) \rightarrow \infty$ as $k \rightarrow \infty$ (see Appendix A).

For any $q \in H^{-1}(\mathbb{T})$ we denote by L_q the Hill operator

$$(2) \quad L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$$

viewed as an operator on the space $H^{-1}(\mathbb{T}_2)$ with domain $\text{Dom}(L_q) = H^1(\mathbb{T}_2)$. The classical spectral theory of Hill's operator can be extended for such singular potentials (see Appendix B). It is proved in Appendix B, Lemma 6, that the spectrum of L_q is discrete, real, and of the form $\mathbf{spec}(L_q) = \{\lambda_0(q) \leq \lambda_1(q) \leq \lambda_2(q) \leq \dots\}$, the corresponding eigenspaces are of finite dimension, and $\lambda_k(q) \rightarrow \infty$ as $k \rightarrow \infty$.

Clearly, for any $r \in L_0^2(\mathbb{T})$, $r^2 - \|r\|^2$ defines a bounded linear functional on $H^1(\mathbb{T})$ satisfying $\langle r^2 - \|r\|^2, 1 \rangle = 0$ where $\|r\|^2 \stackrel{\text{def}}{=} \int_0^1 r^2(x) dx$. Hence, $r^2 - \|r\|^2$ is an element in $H_0^{-1}(\mathbb{T})$ and one can introduce the nonlinear map

$$(3) \quad R : L_0^2(\mathbb{T}) \rightarrow H_0^{-1}(\mathbb{T}), \quad r \mapsto r' + r^2 - \|r\|^2,$$

referred to as the Riccati map. The following result shows how the operators T_r and L_q are related if $q = R(r)$. It can be shown in a straightforward way (see [6]).

Lemma 1. *Let $r \in L_0^2(\mathbb{T})$ and assume that $q \in H_0^{-1}(\mathbb{T})$ satisfies Riccati's equation $q = R(r)$. Then*

- (a) $\mathbf{spec}(T_r) = \|r\|^2 + \mathbf{spec}(L_q)$;
- (b) for any $k \geq 0$, the eigenspaces $V_{\lambda_k}(L_q)$ and $V_{\lambda_k + \|r\|^2}(T_r)$ have the same dimension;
- (c) the $L^2(\mathbb{T})$ -norm of r coincides with the absolute value of the first eigenvalue $\lambda_0(q)$ of Hill's operator L_q , i.e. $\|r\|^2 = |\lambda_0(q)| = -\lambda_0(q)$;
- (d) the first eigenvalue $\lambda_0(q)$ of the operator L_q is simple and the corresponding eigenfunction normalized by $\|f_0\|^2 = 1$ and $f_0(0) > 0$ is $f_0 \stackrel{\text{def}}{=} \rho / \|\rho\|$. Hence, f_0 is in $H^1(\mathbb{T})$, doesn't vanish on \mathbb{T} and satisfies $r = \frac{f_0'}{f_0}$.

Lemma 1 (d) implies the following result:

Corollary 1.. *The Riccati map $R : L_0^2(\mathbb{T}) \rightarrow H_0^{-1}(\mathbb{T})$ is injective. Moreover, if $q \in \text{range}(R)$ then $R^{-1}(q) = f'_0(\cdot, q)/f_0(\cdot, q)$.*

Remark 1.. *Note that the quotient f'_0/f_0 is independent of the normalization of f_0 .*

3. Riccati's map

By definition, Riccati's map $R : L_0^2(\mathbb{T}) \rightarrow H_0^{-1}(\mathbb{T})$ is given by formula (3). For any $\alpha \geq 0$ denote by R_α the restriction of R to $H_0^\alpha(\mathbb{T}) \subset L_0^2(\mathbb{T})$, $R_\alpha \stackrel{\text{def}}{=} R|_{H_0^\alpha(\mathbb{T})}$.

Proposition 1.. *The Riccati map $R_\alpha : H_0^\alpha(\mathbb{T}) \rightarrow H_0^{\alpha-1}(\mathbb{T})$ is a diffeomorphism from $H_0^\alpha(\mathbb{T})$ to $H_0^{\alpha-1}(\mathbb{T})$.*

Proof. Let us consider first the case $\alpha = 0$. By definition, $R_0 = R$. First note that R is continuous. The claimed statement then follows from the following three assertions:

- (i) R has dense image;
- (ii) R is surjective (and hence bijective by Corollary 1);
- (iii) R and R^{-1} are differentiable.

Let $q \in C_0^\infty(\mathbb{T}) \stackrel{\text{def}}{=} \{f \in C^\infty(\mathbb{T}) \mid \int_0^1 f(x)dx = 0\}$ and $\lambda_0(q)$ be the first eigenvalue of L_q . By the classical theory of Hill's equation, the corresponding eigenfunction f_0 doesn't have zeroes (see [15]). Hence, $L_q f_0 = \lambda_0(q) f_0$ can be rewritten as $q - \lambda_0(q) = \frac{f''_0}{f_0} = r' + r^2$ where $r \stackrel{\text{def}}{=} \frac{f'_0}{f_0}$. Integrating the last equality one gets $\lambda_0(q) = -\|r\|^2$, and therefore, $q = r' + r^2 - \|r\|^2 = R(r)$. As $C_0^\infty(\mathbb{T})$ -functions are dense in $H_0^{-1}(\mathbb{T})$, the image of the Riccati map is dense in $H_0^{-1}(\mathbb{T})$ proving item (i).

To prove item (ii) take an arbitrary $q \in H_0^{-1}(\mathbb{T})$. It follows from item (i) that there exists a sequence $\{q_k\}_{k=1}^\infty \subset \text{range}(R)$ such that $q_k \rightarrow q$ ($k \rightarrow \infty$) in $H_0^{-1}(\mathbb{T})$. Consider the sequence $\{r_k\}_{k=1}^\infty \subset L_0^2(\mathbb{T})$ such that $q_k = R(r_k)$. Item (c) of Lemma 1 shows that $\|r_k\|^2 = |\lambda_0(q_k)|$ where $\lambda_0(q_k)$ is the first eigenvalue of the Hill operator $L_{q_k} = -\frac{d^2}{dx^2} + q_k$. As a function of the potential $q \in H_0^{-1}(\mathbb{T})$ the first eigenvalue $\lambda_0(q)$ is continuous on $H_0^{-1}(\mathbb{T})$ – see Appendix B, Lemma 6. Therefore, $\|r_k\| \rightarrow \sqrt{|\lambda_0(q)|}$ as $k \rightarrow \infty$. Hence, the sequence $\{r_k\}_{k=1}^\infty$ is bounded in $H^0(\mathbb{T})$.

Consider the map $S_0 : H^0(\mathbb{T}) \rightarrow H^{-1}(\mathbb{T})$, $r \mapsto r^2$. This map can be viewed as a composition of two maps $S_0 = \iota \circ S_1$, where $S_1 : H^0(\mathbb{T}) \rightarrow H^{-3/4}(\mathbb{T})$ is given

by the formula $r \mapsto r^2$, and $\iota : H^{-3/4}(\mathbb{T}) \hookrightarrow H^{-1}(\mathbb{T})$ is the standard inclusion of Sobolev spaces. By Rellich's theorem ι is compact. As each Fourier coefficient $(\hat{r^2})_n$ of r^2 satisfies $|(\hat{r^2})_n| \leq \|r\|^2$ ($n \in \mathbb{Z}$) it follows that there exists $C > 0$ so that $\|r^2\|_{-3/4} \leq C\|r\|^2$, hence the map S_1 is bounded. Therefore, there exists a subsequence $\{r_{k_j}\}_{j=1}^\infty$ of $\{r_k\}$ and an element $g \in H^{-1}(\mathbb{T})$ such that $r_{k_j}^2 \rightarrow g$ ($j \rightarrow \infty$) in $H^{-1}(\mathbb{T})$. By the definition of Riccati's map, $r'_{k_j} = q_{k_j} - r_{k_j}^2 + \|r_{k_j}\|^2 \in H_0^{-1}(\mathbb{T})$. Each of the terms on the right hand side of the latter equation converges in $H^{-1}(\mathbb{T})$. Hence r'_{k_j} converges to some element $s \in H_0^{-1}(\mathbb{T})$. Denote by r the unique element in $L_0^2(\mathbb{T})$ such that $s = r'$. As $\|r - r_{k_j}\| \leq \text{const} \|r' - r'_{k_j}\|_{-1} \rightarrow 0$ for $j \rightarrow \infty$ and R is continuous it then follows that $q_{k_j} = R(r_{k_j}) \rightarrow R(r)$ ($j \rightarrow \infty$) in $H^{-1}(\mathbb{T})$. Therefore, $q = R(r)$ and claim (ii) is proved.

Towards claim (iii) note that R is continuously differentiable. The corresponding property for R^{-1} follows from the identity $R^{-1}(q) = \frac{f'_0(\cdot, q)}{f_0(\cdot, q)}$ (see Lemma 1 (d)) as follows: the first eigenfunction $f_0(\cdot, q)$ considered as a map $H^{-1}(\mathbb{T}) \rightarrow H^1(\mathbb{T}_2)$ is continuously differentiable (in fact real analytic) in the variable q when normalized so that $\int_0^2 f_0^2(x, q) dx = 2$ and $f_0(0, q) > 0$ (see Appendix B, Lemma 6). Note that any one-periodic function $f \in H^1(\mathbb{T}_2)$ can be isometrically identified with a function in $H^1(\mathbb{T})$. Indeed, it follows from $f(x+1) = f(x)$ that the coefficients \hat{f}_{2k+1} ($k \in \mathbb{Z}$) in the Fourier expansion $f = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{i\pi k x}$ of the function f on the interval $[0, 2]$ all vanish and the condition $f \in H^1(\mathbb{T}_2)$ implies that $f \in H^1(\mathbb{T})$. In particular, we see that the normalized eigenfunction $f_0(\cdot, q)$ considered as a map $H^{-1}(\mathbb{T}) \rightarrow H^1(\mathbb{T})$ is continuously differentiable (in fact real analytic). This shows that R^{-1} is continuously differentiable.

The proof of the proposition for $\alpha > 0$ is straightforward. \square

The following statement is a generalization of the corresponding classical result. Attempts to prove it using the classical approach (see [15]) fail at several stages.

Corollary 2. *For any $q \in H_0^{-1}(\mathbb{T})$, the first eigenvalue $\lambda_0(q)$ of L_q is simple. Any eigenfunction corresponding to $\lambda_0(q)$ is an element in $H^1(\mathbb{T})$ and doesn't vanish on \mathbb{T} . Moreover $\|R^{-1}(q)\|$ is a spectral invariant of L_q .*

Proof. Take $r = R^{-1}(q)$. By Lemma 1 (d) the first eigenvalue $\lambda_0(q)$ of L_q is simple and $f_0(x) = \rho(x)/\|\rho\|$ is an eigenfunction corresponding to $\lambda_0(q)$ where $\rho(x) = \exp(\int_0^x r(v) dv)$. Hence f_0 is an element in $H^1(\mathbb{T})$ and doesn't vanish on \mathbb{T} . It follows from Lemma 1 (c) that $\|R^{-1}(q)\|^2 = -\lambda_0(q)$ which is obviously a spectral invariant. \square

Denote by $H_0^\alpha(\mathbb{T}, \mathbb{C})$ the complexification of the (real) Sobolev space $H_0^\alpha(\mathbb{T})$. For complex-valued functions $r \in H_0^\alpha(\mathbb{T}, \mathbb{C})$, $\alpha \geq 0$, the (complex) Riccati map is

defined by the formula $R_\alpha(r) \stackrel{\text{def}}{=} r' + r^2 - \int_0^1 r^2(x)dx$. Using the same arguments as in the real case one concludes that R_α maps $H_0^\alpha(\mathbb{T}, \mathbb{C})$ into $H_0^{\alpha-1}(\mathbb{T}, \mathbb{C})$ and is an analytic map. As a consequence of Proposition 1 one easily obtains

Theorem 2. *For any $\alpha \geq 0$ there exist open neighborhoods $U \subset H_0^\alpha(\mathbb{T}, \mathbb{C})$ and $W \subset H_0^{\alpha-1}(\mathbb{T}, \mathbb{C})$ of $H_0^\alpha(\mathbb{T})$ and $H_0^{\alpha-1}(\mathbb{T})$ respectively such that the Riccati map $R_\alpha : U \rightarrow W$ is an analytic isomorphism.*

Proof of Theorem 1. The claimed results follow from Theorem 2 and Lemma 1.

4. Applications

This section contains several applications of the Riccati representation of the elements of the Sobolev space $H^{-1}(\mathbb{T})$: we give asymptotic formulas for the spectrum of the Hill operator with a singular potential and an a priori estimate of the potential q in terms of the gap lengths. Analogous asymptotic formulas are proved for the Dirichlet spectrum. Clearly, classical Floquet theory for Hill's operator $-\frac{d^2}{dx^2} + q$ for potentials q in $L^2(\mathbb{T})$ cannot be extended to potentials in $H^{-1}(\mathbb{T})$. Nevertheless we show that some features of the Floquet theory of Hill's operator can be extended without essential changes to the case of singular potentials from the Sobolev spaces $H^{-\alpha}(\mathbb{T})$, $0 < \alpha \leq 1$.

4.1. Periodic spectrum

In this paragraph we prove four results on the spectrum of Hill's operator.

Theorem 3. *The spectrum of Hill's operator $L_q = -\frac{d^2}{dx^2} + q$ on $H^{-1}(\mathbb{T}_2)$ with singular potential $q \in H^{-1}(\mathbb{T})$ is discrete, $\text{spec}(L_q) = \{\lambda_0(q) < \lambda_1(q) \leq \lambda_2(q) < \dots\}$, $\lambda_k(q) \rightarrow \infty$ as $k \rightarrow \infty$. The eigenvalues are totally ordered, $\lambda_{2k-1}(q) \leq \lambda_{2k}(q)$ and $\lambda_{2k}(q) < \lambda_{2k+1}(q)$, where the equality $\lambda_{2k-1}(q) = \lambda_{2k}(q)$ means that the corresponding eigenspace has two dimensions. Otherwise, the corresponding eigenspaces are one-dimensional. The eigenvalues $\lambda_{2k-1}(q) \leq \lambda_{2k}(q)$ with k odd correspond to anti-periodic eigenfunctions, i.e. $f(x+1) = -f(x)$ and those with k even to periodic ones, i.e. $f(x+1) = f(x)$.*

Proof. By Theorem 1 there exists $r \in L_0^2(\mathbb{T})$ such that $q = R(r)$. Theorem 3 then follows from Lemma 1 and the spectral properties of the impedance operator T_r (see Appendix A, §5.1). \square

For any $k \geq 0$, denote by $\gamma_k(q) \stackrel{\text{def}}{=} \lambda_{2k}(q) - \lambda_{2k-1}(q)$ the k 'th gap-length and by $\gamma(q)$ the sequence $\{\gamma_k(q)\}_{k \geq 1}$. The following two theorems are applications of results in [9] concerning the spectrum of the impedance operator T_r for $r \in H_0^0(\mathbb{T})$.

Theorem 4. For any $q \in H_0^{-1}(\mathbb{T})$, $\{\gamma_k(q)\}_{k \geq 1}$ belongs to the sequence space h^{-1} .

Theorem 5. There exists a constant $c > 0$ such that for every potential $q \in H_0^{-1}(\mathbb{T})$

$$(4) \quad \|q\|_{-1} \leq c\|\gamma(q)\|_{-1}(1 + c\|\gamma(q)\|_{-1})^3$$

Proof of Theorems 4 and 5. Take $r \stackrel{\text{def}}{=} R^{-1}(q)$. By Lemma 1, the operators L_q and T_r have, up to a translation, the same spectrum. Hence these operators have the same gap-lengths. Theorem 4 thus follows from Theorem 1.1 in [9].

Using $q = R(r) \stackrel{\text{def}}{=} r' + r^2 - \|r\|^2$, the Cauchy-Schwartz inequality, and the easily verified inequalities $\|r'\|_{-1} \leq \|r\|$ and $\|r^2\|_{-1} \leq c_1\|r\|^2$ for some constant $c_1 > 0$, one concludes that there is a constant $c_2 > 0$ so that for any $r \in L_0^2(\mathbb{T})$, and $q \in H_0^{-1}(\mathbb{T})$ with $q = R(r)$,

$$(5) \quad \|q\|_{-1} = \|R(r)\|_{-1} \leq \|r'\|_{-1} + \|r^2\|_{-1} + \|r\|^2 \leq c_2\|r\|(1 + c_2\|r\|).$$

By Theorem 1.2 in [9] for the impedance operator T_r , there exists $c_3 > 0$ so that for any $r \in L_0^2(\mathbb{T})$ and $q \in H_0^{-1}(\mathbb{T})$ with $q = R(r)$

$$(6) \quad \|r\| \leq c_3\|\gamma(q)\|_{-1}(1 + c_3\|\gamma(q)\|_{-1}).$$

Combining these last two estimates, Theorem 5 follows.

Remark 2. By Lemma 1 (d) and (6) we obtain the following estimate of the first eigenvalue $\lambda_0(q)$ in the terms of the sequence of gap lengths

$$\sqrt{|\lambda_0(q)|} \leq c_3\|\gamma(q)\|_{-1}(1 + c_3\|\gamma(q)\|_{-1}).$$

4.2. Dirichlet spectrum

Consider the operator $L_q^{Dir} = -\frac{d^2}{dx^2} + q$ on $H^{-1}[0, 1] = (H_c^1[0, 1])'$ with $q \in H^{-1}(\mathbb{T})$ and domain $\text{Dom}(L_q^{Dir}) = H_{Dir}^1[0, 1]$ (see Appendix B, §6.2.). First we need some auxiliary results which again can be proved in a straightforward way.

Lemma 2. For given $q \in H_0^{-1}(\mathbb{T})$, let $r \stackrel{\text{def}}{=} R^{-1}(q) \in L_0^2(\mathbb{T})$ where R^{-1} is the inverse of Riccati's map. Then

$$(a) \quad \text{spec}(L_q^{Dir}) = \text{spec}(T_r^{Dir}) - \|r\|^2;$$

(b) for any $k \geq 1$, the eigenspaces $V_{\mu_k}(L_q^{Dir})$ and $V_{\mu_k + \|r\|^2}(T_r^{Dir})$ have the same dimension.

The next two theorems generalize results of [17].

Theorem 6. *The spectrum of L_q^{Dir} is discrete $\text{spec}(L_q^{Dir}) = \{\infty < \mu_1(q) < \mu_2(q) < \dots\}$, the corresponding eigenspaces are one-dimensional, and $\mu_k \rightarrow \infty$ as $k \rightarrow \infty$.*

Proof. The claimed results follow from the spectral properties of the operator T_r^{Dir} (see Appendix A, §5.2.) and Lemma 2. \square

Theorem 7. *A sequence $\{\infty < \sigma_1 < \sigma_2 < \dots\}$, $\sigma_k \rightarrow \infty$ as $k \rightarrow \infty$, is the Dirichlet spectrum of L_q^{Dir} for some $q \in H^{-1}(\mathbb{T})$, if and only if $\sigma_k = \text{const} + (k\pi + s_k)^2$ where $\{s_k\}_{k \geq 1} \in h^0$.*

Before proving Theorem 7 let us state Theorem 8. Given $q \in H_0^{-1}(\mathbb{T})$ let $r \stackrel{\text{def}}{=} R^{-1}(q)$. The eigenvalues $\tilde{\mu}_k$ of T_r^{Dir} satisfy $0 < \tilde{\mu}_1 < \tilde{\mu}_2 < \dots$ and, according to Lemma 2, $\tilde{\mu}_k = \mu_k + \|r\|^2$ where $\mu_{k \geq 1}$ is the spectrum of L_q^{Dir} . Introduce the sequence $\omega(q) \stackrel{\text{def}}{=} \{(\ln(\tilde{\mu}_k^{1/2}))\}_{k \geq 1}$. Unlike the eigenfunctions of L_q^{Dir} , an eigenfunction \tilde{g}_k of the operator T_r^{Dir} corresponding to the eigenvalue $\tilde{\mu}_k = \mu_k + \|r\|^2$ is in $H^2[0, 1]$. Hence we can define $\kappa(q) \stackrel{\text{def}}{=} \{\kappa_k(q)\}$, $\kappa_k(q) \stackrel{\text{def}}{=} \ln \left| \frac{\tilde{g}'_k(1)}{\tilde{g}'_k(0)} \right|$. Note that $\kappa_k(q)$ is independent of the normalization of \tilde{g}_k and for q in $L_0^2(\mathbb{T})$ coincides with $\ln |g'_k(1)/g'_k(0)|$ where $g_k \in H^2[0, 1]$ is an eigenfunction of L_q^{Dir} corresponding to the eigenvalue μ_k . Let us consider the map

$$(7) \quad H_0^{-1}(\mathbb{T}) \ni q \mapsto (\omega(q), \kappa(q)).$$

To discuss its properties introduce

$$S \stackrel{\text{def}}{=} \{ \{ \ln(k\pi + s_k) \}_{k \geq 1} \mid \{s_k\}_{k \geq 1} \in h^0, 0 < k\pi + s_k < (k+1)\pi + s_{k+1} \}.$$

Theorem 8. *The mapping $q \mapsto (\mu(q), \kappa(q))$ is a real analytic isomorphism onto $S \times h^0$.*

Proof of Theorem 7 and 8. The stated results follow directly from Corollary 5.5 and Corollary 5.6 in [2] together with Lemma 2 and Theorem 2.

4.3. Discriminant of Hill's operator

The potentials $q \in H_0^{-1}(\mathbb{T})$ are too singular for L_q to admit fundamental solutions. Hence the Floquet matrix cannot be defined in this situation. However, it turns out that the trace $\Delta(\lambda, q)$ of the Floquet matrix, referred to as discriminant, can still be defined as we will explain now. Recall that the discriminant $\tilde{\Delta}$ is defined

for $\tilde{\lambda} \in \mathbb{C}$ and $r \in L_0^2(\mathbb{T})$ arbitrary, by $\tilde{\Delta}(\tilde{\lambda}, r) \stackrel{\text{def}}{=} u_1(1, \tilde{\lambda}, r) + u'_2(1, \tilde{\lambda}, r)$ where $u_1(x, \tilde{\lambda}, r)$ and $u_2(x, \tilde{\lambda}, r)$ are the fundamental solutions of the equation

$$(8) \quad -u'' - 2ru' = \tilde{\lambda}u.$$

For $q \in L_0^2(\mathbb{T})$ the discriminant $\Delta(\lambda, q)$ is well defined and related to $\tilde{\Delta}(\tilde{\lambda}, r)$ as follows. Define $r \stackrel{\text{def}}{=} R^{-1}(q)$ and $\rho(x) \stackrel{\text{def}}{=} \exp(\int_0^x r(v)dv)$. As $q \in L^2(\mathbb{T})$, the equation

$$(9) \quad -y'' + qy = \lambda y$$

admits fundamental solutions $y_1(x, \lambda, q)$ and $y_2(x, \lambda, q)$, i.e. solutions satisfying the initial conditions $y_1(0, \lambda, q) = y'_2(0, \lambda, q) = 1$ and $y'_1(0, \lambda, q) = y_2(0, \lambda, q) = 0$. It is easy to see that for $j = 1, 2$, the functions $\tilde{y}_j(x, \tilde{\lambda}, r) \stackrel{\text{def}}{=} y_j(x, \lambda, q)/\rho(x)$ are solutions of (8) with $\tilde{\lambda} = \lambda + \|r\|^2$ and $\|r\|^2 = \int_0^1 r^2(x)dx$. As $\rho(0) = 1$, one obtains $\tilde{y}_1(0, \tilde{\lambda}, r) = \tilde{y}'_2(0, \tilde{\lambda}, r) = 1$, $\tilde{y}_2(0, \tilde{\lambda}, r) = 0$, and $\tilde{y}'_1(0, \tilde{\lambda}, r) = -\rho'(0)$. Hence, the fundamental solutions u_1 and u_2 of equation (8) are related to \tilde{y}_1 and \tilde{y}_2 by

$$u_1(x, \tilde{\lambda}, r) = \tilde{y}_1(x, \tilde{\lambda}, r) + \rho'(0)\tilde{y}_2(x, \tilde{\lambda}, r)$$

and

$$u_2(x, \tilde{\lambda}, r) = \tilde{y}_2(x, \tilde{\lambda}, r).$$

Using that $\rho(1) = \rho(0) = 1$ and $\rho'(0) = \rho'(1)$ we obtain

$$\begin{aligned} \tilde{\Delta}(\tilde{\lambda}, r) &\stackrel{\text{def}}{=} u_1(1, \tilde{\lambda}, r) + u'_2(1, \tilde{\lambda}, r) \\ &= (y_1/\rho + \rho'(0)y_2/\rho)|_{(x=1, \lambda, q)} + (y'_2/\rho - y_2\rho'/\rho^2)|_{(x=1, \lambda, q)} \\ (10) \quad &= y_1(1, \lambda, q) + y'_2(1, \lambda, q) \stackrel{\text{def}}{=} \Delta(\lambda, q). \end{aligned}$$

Hence, we can define $\Delta(\lambda, q)$ for $q \in H_0^{-1}(\mathbb{T})$ by the latter identity.

Definition 1. For any $q \in H_0^{-1}(\mathbb{T})$ and $\lambda \in \mathbb{C}$,

$$(11) \quad \Delta(\lambda, q) \stackrel{\text{def}}{=} \tilde{\Delta}(\lambda + \|r\|^2, r)$$

where $r = R^{-1}(q)$, $\|r\|^2 = \int_0^1 r^2(x)dx$.

It follows directly from Theorem 2 and the properties of $\tilde{\Delta}(\tilde{\lambda}, r)$ that $\Delta(\lambda, q)$ is an analytic function on $\mathbb{C} \times W$ where $W \subset H_0^{-1}(\mathbb{T}, \mathbb{C})$ is the neighborhood of $H_0^{-1}(\mathbb{T})$ given by Theorem 2, and that the zeroes of $\Delta(\lambda, q)^2 - 4$ are precisely the eigenvalues of L_q , i.e. for any $\lambda \in \mathbb{C}$,

$$\Delta(\lambda, q)^2 - 4 = 0 \text{ if and only if } \lambda \in \text{spec}(L_q).$$

4.4. Isospectral invariance of Riccati's map

For any $q \in H_0^{-1}(\mathbb{T})$ denote by $\text{Iso}(L_q)$ the set of potentials $p \in H_0^{-1}(\mathbb{T})$ such that $\text{spec}(L_p) = \text{spec}(L_q)$, i.e.

$$\text{Iso}(L_q) \stackrel{\text{def}}{=} \{p \in H_0^{-1}(\mathbb{T}) \mid \text{spec}(L_p) = \text{spec}(L_q)\}.$$

Similarly for any $r \in L_0^2(\mathbb{T})$, denote

$$\text{Iso}(T_r) \stackrel{\text{def}}{=} \{u \in L_0^2(\mathbb{T}) \mid \text{spec}(T_u) = \text{spec}(T_r)\}.$$

Theorem 9. *For every $r \in L_0^2(\mathbb{T})$*

$$R(\text{Iso}(T_r)) = \text{Iso}(L_{R(r)}).$$

Proof. Let $r \in L_0^2(\mathbb{T})$ and $q \stackrel{\text{def}}{=} R(r) \in H_0^{-1}(\mathbb{T})$. To see that $\text{Iso}(L_{R(r)}) \subset R(\text{Iso}(T_r))$ take any $p \in \text{Iso}(L_q)$ and set $u \stackrel{\text{def}}{=} R^{-1}(p)$. By Lemma 1 (a) and (d),

$$\text{spec}(T_u) = -\lambda_0(p) + \text{spec}(L_p) = -\lambda_0(q) + \text{spec}(L_q) = \text{spec}(T_r).$$

Conversely, take $u \in \text{Iso}(T_r)$ and let $p \stackrel{\text{def}}{=} R(u)$. By the definition of the isospectral set $\text{Iso}(T_r)$ we obtain that $\text{spec}(T_u) = \text{spec}(T_r)$. It follows from Lemma 1 (a) that $\text{spec}(L_p) = \text{spec}(T_u) - \|u\|^2$ and $\text{spec}(L_q) = \text{spec}(T_r) - \|r\|^2$. By Corollary 4 (Appendix A) the L^2 -norm $\|r\|$ of the potential $r \in L_0^2(\mathbb{T})$ is a spectral invariant of the impedance operator T_r . Hence, it follows from $\text{spec}(T_u) = \text{spec}(T_r)$, that $\|u\|^2 = \|r\|^2$ and therefore $\text{spec}(L_p) = \text{spec}(L_q)$. This completes the proof of Theorem 9. \square

Corollary 3. *For every potential $q \in H_0^{-1}(\mathbb{T})$, the isospectral set $\text{Iso}(L_q)$ is compact in $H_0^{-1}(\mathbb{T})$.*

Proof. First we prove that for any $r \in L_0^2(\mathbb{T})$ the isospectral set $\text{Iso}(T_r)$ is compact. Let $\{r_k\}_{k \geq 0}$ be a sequence in $\text{Iso}(T_r)$. It follows from the spectral invariance of the L^2 -norm of r that $\|r_k\| = \|r\|$ for any $k \geq 1$ (Corollary 4). Hence, there exists a subsequence $\{r_{k_j}\}_{j \geq 1}$ that converges weakly to some element $u \in L_0^2(\mathbb{T})$. By Lemma 2.4 in [9], the sequence of discriminants $\tilde{\Delta}(\tilde{\lambda}, r_k)$ of T_{r_k} converges to $\tilde{\Delta}(\tilde{\lambda}, u)$ as $k \rightarrow \infty$ uniformly on bounded subsets of \mathbb{C} . On the other side, the spectral invariance of r_k shows that $\tilde{\Delta}(\tilde{\lambda}, r_k) = \tilde{\Delta}(\tilde{\lambda}, r)$ and therefore $\tilde{\Delta}(\tilde{\lambda}, r) = \tilde{\Delta}(\tilde{\lambda}, u)$. In particular, $\text{spec}(T_r) = \text{spec}(T_u)$ and $\|u\| = \|r\|$. As r_{k_j} converges weakly to u , $\|u - r_{k_j}\|^2 = 2\|u\|^2 - 2(u, r_{k_j}) \rightarrow 0$ as $j \rightarrow \infty$. Therefore, the isospectral set $\text{Iso}(T_r)$ is compact.

Theorem 9 and the continuity of the Riccati map $R : L_0^2(\mathbb{T}) \rightarrow H^{-1}(\mathbb{T})$ then imply the compactness of the isospectral sets $\text{Iso}(L_q)$, $q \in H_0^{-1}(\mathbb{T})$. \square

4.5. Complex potentials

In a straightforward way many of the previous results can be extended for complex potentials in some open neighborhood W of $H_0^{-1}(\mathbb{T})$ in $H_0^{-1}(\mathbb{T}, \mathbb{C})$. As an example we mention the following theorem which can be proved using the same arguments as in the proof of Lemma 1.

Denote by U and W the neighborhoods given by Theorem 2.

Theorem 10. *For given $q \in W \subset H_0^{-1}(\mathbb{T}, \mathbb{C})$, let $r \stackrel{\text{def}}{=} R^{-1}(q) \in U \subset L_0^2(\mathbb{T}, \mathbb{C})$ where R^{-1} is the inverse of Riccati's map $R : U \rightarrow W$. Then*

- (a) $\text{spec}(L_q) = \text{spec}(T_r) - \int_0^1 r(x)^2 dx$;
- (b) *for any $k \geq 0$, the eigenspaces $V_{\lambda_k}(L_q)$ and $V_{\tilde{\lambda}_k}(T_r)$ have the same dimension where $\tilde{\lambda}_k \stackrel{\text{def}}{=} \lambda_k + \int_0^1 r(x)^2 dx$.*

In view of Theorem 10 one can reformulate results on the spectrum of the impedance operator T_r with $r \in U$ in terms of the corresponding result for the operator L_q with $q = R(r) \in W$, and vice versa.

A complex analogue of Lemma 2 can be easily proved as well.

5. Appendix A: Impedance operator

An impedance operator is a Sturm-Liouville operator of a special type and is treated in numerous articles and books – see [13, 1, 2, 9]. For the convenience of the reader we recall those properties needed in the main part of this paper.

5.1. Periodic problem

For any $r \in L_0^2(\mathbb{T})$ denote by ρ the element in $H^1(\mathbb{T})$ satisfying $\rho' = r\rho$ and $\rho(0) = 1$. Then ρ is a one-periodic, absolutely continuous, positive function given by $\rho(x) = \exp(\int_0^x r(v)dv)$. The periodic impedance operator T_r is defined on the Hilbert space $L^2(\mathbb{T}_2)$ with domain $\text{Dom}(T_r) = H^2(\mathbb{T}_2)$ by the formula

$$(12) \quad T_r(u) \stackrel{\text{def}}{=} -(\rho^2 u')' / \rho^2 = -u'' - 2ru'.$$

Note that the operator T_r is positive and symmetric with respect to the $L^2(\mathbb{T}_2)$ -inner product $(f, g)_\rho \stackrel{\text{def}}{=} \int_0^2 fg\rho^2 dx$.

As T_r has compact resolvent the spectrum of T_r is discrete. It turns out (see [13, 12]) that $\text{spec}(T_r)$ is of the form $\text{spec}(T_r) = \{0 = \tilde{\lambda}_0(r) < \tilde{\lambda}_1(r) \leq \tilde{\lambda}_2(r) \leq \dots\}$, the corresponding eigenspaces are of dimension 1 or 2, and $\tilde{\lambda}_k(r) \rightarrow \infty$ as $k \rightarrow \infty$. For $k \geq 0$ even, the eigenfunctions \tilde{f}_{2k-1} and \tilde{f}_{2k} are periodic while for k

odd, \tilde{f}_{2k-1} and \tilde{f}_{2k} are anti-periodic. If $\lambda_{2k-1}(r) = \lambda_{2k}(r)$ then the corresponding eigenvalue has multiplicity 2 (cf. [12]). Indeed, the impedance operator T_r can be transformed by a change of the variable $y = y(x) \stackrel{\text{def}}{=} \int_0^x (1/\rho^2(s)) ds$ to the linear operator $-\rho^{-4} \frac{d^2}{dy^4}$ on the torus \mathbb{T}_l with period $l \stackrel{\text{def}}{=} y(1)$ whose spectral properties are established in [12].

Lemma 3. *The first eigenvalue $\tilde{\lambda}_0(r) = 0$ of T_r is simple and the corresponding eigenspace is spanned by the constant function $\tilde{f}_0 \stackrel{\text{def}}{=} 1/||\rho||$.*

Proof. Assume that $u \in \ker(T_r)$. Integrating by parts we obtain

$$(13) \quad 0 = (T_r(u), u)_\rho \stackrel{\text{def}}{=} - \int_0^2 (\rho^2 u')' u dx = \int_0^2 \rho^2 (u')^2 dx.$$

As ρ is positive, $u' \equiv 0$ and hence u is constant. This shows that the dimension of $\ker(T_r)$ is equal to one and Lemma 3 is proved. \square

The discriminant $\tilde{\Delta}$ of the impedance operator T_r is defined for $\tilde{\lambda} \in \mathbb{C}$ and $r \in L_0^2(\mathbb{T})$ arbitrary, by $\tilde{\Delta}(\tilde{\lambda}, r) \stackrel{\text{def}}{=} u_1(1, \tilde{\lambda}, r) + u_2'(1, \tilde{\lambda}, r)$ where $u_1(x, \tilde{\lambda}, r)$ and $u_2(x, \tilde{\lambda}, r)$ are the fundamental solutions of the equation $-u'' - 2ru' = \tilde{\lambda}u$. It follows from the results in [1] that the discriminant $\tilde{\Delta}(\tilde{\lambda}, r)$ is an analytic function on $\mathbb{C} \times L_0^2(\mathbb{T})$. The following lemma can be proved in straightforward way using the results in [1].

Lemma 4. *The discriminant function $\tilde{\Delta}(\tilde{\lambda}, r)$ is a spectral invariant of the impedance operator T_r . The set of zeroes $\tilde{\lambda}_k$ of the equation $\tilde{\Delta}(\tilde{\lambda}, r)^2 = 4$, counted with their multiplicities, coincides with the spectrum of T_r .*

Lemma 4 together with Corollary 1.2 in [10] then lead to the following

Corollary 4. *The L^2 -norm $||r||$ of $r \in L_0^2(\mathbb{T})$ is a spectral invariant of the impedance operator T_r .*

5.2. Dirichlet problem

The Dirichlet problem for the impedance operator has been considered by many authors – see e.g. [1, 2]. The operator T_r^{Dir} is defined on $L^2[0, 1]$ with domain $\text{Dom}(T_r^{Dir}) = H_{Dir}^2[0, 1] \stackrel{\text{def}}{=} \{f \in H^2[0, 1] \mid f(0) = f(1) = 0\}$. By definition, the operator T_r^{Dir} acts on elements $u \in H_{Dir}^2[0, 1]$ by the formula $T_r^{Dir}(u) \stackrel{\text{def}}{=} -(\rho^2 u')'/\rho^2 = -u'' - 2ru'$ where $\rho(x) \stackrel{\text{def}}{=} \exp(\int_0^x r(v) dv)$. The spectrum

$\text{spec}(T_r^{Dir})$ is called *Dirichlet spectrum* of T_r . It is known that $\text{spec}(T_r^{Dir})$ is discrete, all eigenvalues are simple, and $\text{spec}(T_r^{Dir}) = \{0 < \mu_1(r) < \mu_2(r) < \dots\}$ where $\mu_k(r) \rightarrow \infty$ as $k \rightarrow \infty$. Other spectral properties of T_r^{Dir} , including the solution of an inverse problem, were established in [1, 2]. We refer the reader to these papers.

6. Appendix B: Schrödinger operator

6.1. Periodic problem

Take $q \in H^{-1}(\mathbb{T}) \stackrel{\text{def}}{=} (H^1(\mathbb{T}))'$ and consider Hill's operator

$$(14) \quad L_q = -\frac{d^2}{dx^2} + q$$

on $H^{-1}(\mathbb{T}_2)$ with domain $\text{Dom}(L_q) = H^1(\mathbb{T}_2)$. The elements of $H^{-1}(\mathbb{T})$ can be considered as elements of $H^{-1}(\mathbb{T}_2)$ as follows: to any element $u \in H^{-1}(\mathbb{T})$ with Fourier expansion $u = \sum_{k \in \mathbb{Z}} \hat{u}_k e^{i2k\pi x}$, we assign the unique element in $H^{-1}(\mathbb{T}_2)$ given by the Fourier series $\sum_{k \in \mathbb{Z}} \hat{s}_k e^{ik\pi x}$ with $\hat{s}_{2k} \stackrel{\text{def}}{=} \hat{u}_k$ and $\hat{s}_{2k+1} \stackrel{\text{def}}{=} 0$. The operator L_q acts on elements $u \in H^1(\mathbb{T}_2)$ by $L_q u \stackrel{\text{def}}{=} -u'' + qu$ where the multiplication qu is viewed as an element of $H^{-1}(\mathbb{T}_2)$ according to the formula $\langle qu, v \rangle \stackrel{\text{def}}{=} \langle q, uv \rangle$. The brackets $\langle \cdot, \cdot \rangle$ denote the dual pairing between the elements of $H^{-1}(\mathbb{T}_2) \stackrel{\text{def}}{=} (H^1(\mathbb{T}_2))'$ and $H^1(\mathbb{T}_2)$. As the multiplication map $H^1(\mathbb{T}_2) \times H^1(\mathbb{T}_2) \rightarrow H^1(\mathbb{T}_2)$ given by $u \cdot v \stackrel{\text{def}}{=} uv$ is continuous the linear functional qu is continuous as well. It can be easily seen that L_q induces a bounded operator $L_q : H^1(\mathbb{T}_2) \rightarrow H^{-1}(\mathbb{T}_2)$. Considered as an operator on $H^{-1}(\mathbb{T}_2)$, with domain $\text{Dom}(L_q) = H^1(\mathbb{T}_2)$, L_q is an unbounded operator.

The operators L_q with singular potentials $q \in H^{-\alpha}(\mathbb{T}_2)$, $0 < \alpha \leq 1$, have been considered in [16, 3]. In order to make this paper self-contained we review the case $\alpha = 1$ treated in [16] and prove the auxiliary facts used in the main part of the paper. In our presentation, we mainly follow [16], §1.5.1.

For $M > 0$, $r > 0$, and $n \in \mathbb{N}$ introduce the sets

$$(15) \quad \text{Ext}_M \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) \leq |\text{Im}(\lambda)| - M\}$$

$$(16) \quad \text{Vert}_n(r) \stackrel{\text{def}}{=} \{\lambda = n^2\pi^2 + z \in \mathbb{C} \mid |\text{Re}(z)| \leq n\pi^2, |z| \geq r\}.$$

Via the Fourier transform, we identify L_q with the operator \hat{L}_v on h^{-1} with domain $\text{Dom}(\hat{L}_v) = h^1$. The operator \hat{L}_v acts on the sequences $x = \{x_k\}_{k \in \mathbb{Z}} \in h^1$ by $D + V$ where D and V are the infinite matrices $D \stackrel{\text{def}}{=} (k^2\pi^2\delta_{kl})_{k,l \in \mathbb{Z}}$ and

$V \stackrel{\text{def}}{=} (v(k-l))_{k,l \in \mathbb{Z}}$, δ_{kl} is the Kronecker delta and $v(k)$ are the Fourier coefficients of the potential $q \in H^{-1}(\mathbb{T})$ viewed as an element in $H^{-1}(\mathbb{T}_2)$. The proof of the following result can be found in [16].

Lemma 5. *For every $v \in h^{-1}$ there exist a neighborhood $U(v) \subset h^{-1}$ of v and constants $M > 0$ and $n_0 \in \mathbb{N}$ such that for every $u \in U(v)$, the sets Ext_M and $\text{Vert}_n(n)$ ($n > n_0$) are contained in the resolvent set $\text{resol}(\hat{L}_u)$ of \hat{L}_u . The resolvent $(\lambda \mathbf{1} - \hat{L}_u)^{-1} \in \mathcal{L}(h^{-1}, h^1)$, considered as a function of (λ, u) on $\text{Ext}_M \times U(v)$ or $\text{Vert}_n(n) \times U(v)$ with $n > n_0$, is continuous in (λ, u) and for every $u \in U(v)$ holomorphic in λ . Moreover, for any smooth contour $\Gamma \subset \text{Ext}_M \cup \bigcup_{n > n_0} \text{Vert}_n(n)$ and integer $l \geq 0$, the operator $Q_\Gamma^l(q) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_\Gamma \lambda^l (\lambda \mathbf{1} - \hat{L}_u)^{-1} d\lambda \in \mathcal{L}(h^{-1}, h^1)$ is analytic when viewed as a map $U(v) \rightarrow \mathcal{L}(h^{-1}, h^1)$.*

Lemma 6. (i) *For any $q \in H^{-1}(\mathbb{T})$, L_q has a compact resolvent.*

(ii) *The spectrum of the Hill operator L_q with potential $q \in H^{-1}(\mathbb{T})$ is discrete, $\text{spec}(L_q) = \{\lambda_0(q) \leq \lambda_1(q) \leq \lambda_2(q) \leq \dots\}$, the corresponding eigenspaces are of finite dimension, and $\lambda_k(q) \rightarrow \infty$ as $k \rightarrow \infty$.*

(iii) *As functions of the potential $q \in H^{-1}(\mathbb{T})$, the k 'th eigenvalue $\lambda_k(q) : H^{-1}(\mathbb{T}) \rightarrow \mathbb{R}$, $q \mapsto \lambda_k(q)$ is continuous.*

(iv) *Suppose that the eigenvalue $\lambda_k(q)$ is simple for some $q \in H^{-1}(\mathbb{T})$. Then there exists a neighborhood $U(q) \subset H^{-1}(\mathbb{T})$ of q such that for any $u \in U(q)$, the k -th eigenvalue $\lambda_k(u)$ is simple and the corresponding eigenfunction $f_k(\cdot, u)$ (normalized so that $\int_0^2 f_k^2(x, u) dx = 2$ and $f(x_0, u) > 0$, where $x_0 \in [0, 2]$ is chosen so that $f_k(x_0, q) > 0$ for the given potential q) is analytic as a map $U(q) \rightarrow H^1(\mathbb{T}_2)$.*

Remark 3. *We improve on Lemma 6 in Section 4. (see Theorem 3).*

Remark 4. *An analogue of Lemma 6 is true for potentials $q \in H^{\alpha-1}(\mathbb{T})$ with arbitrary $\alpha \geq 0$. In particular, the eigenvalues $\lambda_k(q)$ are continuous with respect to the norm in $H^{\alpha-1}(\mathbb{T})$. If $\lambda_k(q)$ is simple, the corresponding normalized eigenfunction $f_k(\cdot, q)$ is analytic as a map from a neighborhood of q in $H^{\alpha-1}(\mathbb{T})$ to $H^{\alpha+1}(\mathbb{T}_2)$. The proofs are the same as in the case $\alpha = 0$.*

Essentially the same arguments prove the following extension of Lemma 6 (ii).

Theorem 11. *The spectrum of Hill's operator $L_q = -\frac{d^2}{dx^2} + q$ on $H^{-1}(\mathbb{T}_2, \mathbb{C})$ with singular potential $q \in H^{-1}(\mathbb{T}, \mathbb{C})$ is discrete, $\text{spec}(L_q) = \{\text{Re}(\lambda_0(q)) \leq \text{Re}(\lambda_1(q)) \leq \text{Re}(\lambda_2(q)) \leq \dots\}$, the corresponding eigenspaces are of finite dimension, and $\text{Re}(\lambda_k(q)) \rightarrow \infty$ as $k \rightarrow \infty$.*

6.2. Dirichlet problem

In this subsection we set up the Dirichlet problem for the operator $L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$ on $[0, 1]$ with potential $q \in H^{-1}(\mathbb{T})$. In order to make our presentation self-contained, we give some definitions and auxiliary facts on this problem. Further details (including the case $q \in H^{-\alpha}(\mathbb{T})$, $\alpha < 1$) can be found in [3], §3.

Define the operator L_q^{Dir} on the Sobolev space $H^{-1}[0, 1]$ with domain $\text{Dom}(L_q^{Dir}) = H_{Dir}^1[0, 1]$. By definition, $H_{Dir}^1[0, 1] \stackrel{\text{def}}{=} \{f \in H^1[0, 1] \mid f(0) = f(1) = 0\}$ and $H^{-1}[0, 1] \stackrel{\text{def}}{=} (H_{Dir}^1[0, 1])'$ – see Section 1. where the definition of the Sobolev space $H^1[0, 1]$ is recalled. In a natural way, the elements of $H_{Dir}^1[0, 1]$ can be identified with elements in $H^1(\mathbb{T})$, the corresponding inclusion map $\iota : H_{Dir}^1[0, 1] \rightarrow H^1(\mathbb{T})$ being continuous. For any $u \in H_{Dir}^1[0, 1]$, qu is defined to be the functional in $H^{-1}[0, 1] \stackrel{\text{def}}{=} (H_{Dir}^1[0, 1])'$ given by the formula

$$\langle qu, v \rangle \stackrel{\text{def}}{=} \langle q, \iota(u)\iota(v) \rangle.$$

The mapping $H^{-1}[0, 1] \times H_{Dir}^1[0, 1] \rightarrow H^{-1}[0, 1]$, $(q, u) \mapsto qu$, is continuous as is the operator $\frac{d^2}{dx^2} : H^1[0, 1] \rightarrow H^{-1}[0, 1]$. For any $u \in H_{Dir}^1[0, 1]$ we set $L_q^{Dir}u \stackrel{\text{def}}{=} -\frac{d^2u}{dx^2} + qu$. In this way, L_q^{Dir} is a bounded operator $L_q^{Dir} : H_{Dir}^1[0, 1] \rightarrow H^{-1}[0, 1]$. Considered as an operator on $H^{-1}[0, 1]$, L_q^{Dir} is an unbounded operator.

Lemma 7. *The operator L_q^{Dir} has compact resolvent. As a consequence, the spectrum of L_q^{Dir} is discrete, the eigenspaces are of finite dimension, and in every compact set $K \subset \mathbb{C}$ there are finitely many eigenvalues.*

Proof. As in §3.2 in [3] we identify the operator L_q^{Dir} with an operator on an appropriate sequence space. Then Lemma 7 can be proved using the same arguments as in the proof of Lemma 6. \square

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